St Andrew’s Academy

Mathematics Department

Higher Mathematics

EXPONENTIALS & LOGARITHMS

Exponentials and Logarithms

Contents

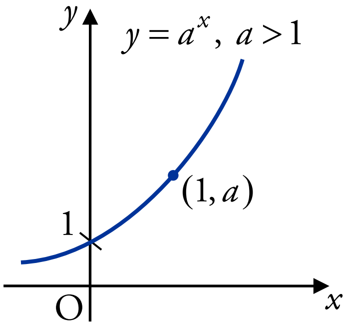
Exponentials and Logarithms

1 Exponentials EF

We have already met exponential functions in the notes on Functions and Graphs..

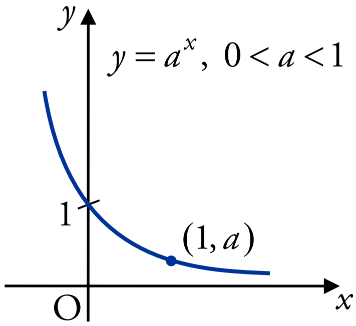
A function of the form $f(x) = a^x$, where $a > 0$ is a constant, is known as an **exponential function** to the base a.

If $a > 1$ then the graph looks like this:



This is sometimes called a **growth** function.

If $0 < a < 1$ then the graph looks like this:



This is sometimes called a **decay** function.

Remember that the graph of an exponential function $f(x) = a^x$ always passes through $(0, 1)$ and $(1, a)$ since:

$$f(0) = a^0 = 1, \qquad\qquad f(1) = a^1 = a.$$

EXAMPLES

1. The otter population on an island increases by 16% per year. How many full years will it take the population to double?

Let u_0 be the initial population.

$$\begin{aligned} u_1 &= 1{\cdot}16u_0 \quad \text{(116\% as a decimal)} \\ u_2 &= 1{\cdot}16u_1 = 1{\cdot}16\left(1{\cdot}16u_0\right) = 1{\cdot}16^2 u_0 \\ u_3 &= 1{\cdot}16u_2 = 1{\cdot}16\left(1{\cdot}16^2 u_0\right) = 1{\cdot}16^3 u_0 \\ &\ \vdots \\ u_n &= 1{\cdot}16^n u_0. \end{aligned}$$

For the population to double after n years, we require $u_n \geq 2u_0$.

We want to know the smallest n which gives $1{\cdot}16^n$ a value of 2 or more, since this will make u_n at least twice as big as u_0.

Try values of n until this is satisfied.

If $n = 2,\ 1{\cdot}16^2 = 1{\cdot}35 < 2$

If $n = 3,\ 1{\cdot}16^3 = 1{\cdot}56 < 2$

If $n = 4,\ 1{\cdot}16^4 = 1{\cdot}81 < 2$

If $n = 5,\ 1{\cdot}16^5 = 2{\cdot}10 > 2$

On a calculator: $\boxed{1}\ \boxed{.}\ \boxed{1}\ \boxed{6}\ \boxed{=}$

$\boxed{1}\ \boxed{.}\ \boxed{1}\ \boxed{6}\ \boxed{\text{ANS}}\ \boxed{=}$

$\boxed{=}$

$\vdots$

Therefore after 5 years the population will double.

2. The efficiency of a machine decreases by 5% each year. When the efficiency drops below 75%, the machine needs to be serviced.
After how many years will the machine need to be serviced?

Let u_0 be the initial efficiency.

$$\begin{aligned} u_1 &= 0{\cdot}95u_0 \quad \text{(95\% as a decimal)} \\ u_2 &= 0{\cdot}95u_1 = 0{\cdot}95\left(0{\cdot}95u_0\right) = 0{\cdot}95^2 u_0 \\ u_3 &= 0{\cdot}95u_2 = 0{\cdot}95\left(0{\cdot}95^2 u_0\right) = 0{\cdot}95^3 u_0 \\ &\ \vdots \\ u_n &= 0{\cdot}95^n u_0. \end{aligned}$$

When the efficiency drops below $0{\cdot}75u_0$ (75% of the initial value) the machine must be serviced. So the machine needs serviced after n years if $0{\cdot}95^n \leq 0{\cdot}75$.

Try values of n until this is satisfied:

$$\begin{aligned}
&\text{If } n=2,\ 0{\cdot}95^2 = 0{\cdot}903 \quad > 0{\cdot}75\\
&\text{If } n=3,\ 0{\cdot}95^3 = 0{\cdot}857 \quad > 0{\cdot}75\\
&\text{If } n=4,\ 0{\cdot}95^4 = 0{\cdot}815 \quad > 0{\cdot}75\\
&\text{If } n=5,\ 0{\cdot}95^5 = 0{\cdot}774 \quad > 0{\cdot}75\\
&\text{If } n=6,\ 0{\cdot}95^6 = 0{\cdot}735 \quad < 0{\cdot}75
\end{aligned}$$

Therefore after 6 years, the machine will have to be serviced.

2 Logarithms EF

Having previously defined what a logarithm is (see the notes on Functions and Graphs) we now look in more detail at the properties of these functions.

The relationship between logarithms and exponentials is expressed as:

$$y = \log_a x \iff x = a^y \qquad \text{where } a,\ x > 0\,.$$

Here, y is the power of a which gives x.

EXAMPLES

1. Write $5^3 = 125$ in logarithmic form.

$5^3 = 125 \iff 3 = \log_5 125\,.$

2. Evaluate $\log_4 16\,.$

The power of 4 which gives 16 is 2, so $\log_4 16 = 2\,.$

3 Laws of Logarithms EF

There are three laws of logarithms which you must know.

Rule 1

$$\log_a x + \log_a y = \log_a(xy) \qquad \text{where } a, x, y > 0\,.$$

If two logarithmic terms with the same base number (a above) are being added together, then the terms can be combined by multiplying the arguments (x and y above).

EXAMPLE

1. Simplify $\log_5 2 + \log_5 4\,.$

$$\begin{aligned}
&\log_5 2 + \log_5 4\\
&= \log_5(2\times 4)\\
&= \log_5 8.
\end{aligned}$$

Rule 2

$$\log_a x - \log_a y = \log_a\left(\tfrac{x}{y}\right) \qquad \text{where } a, x, y > 0.$$

If a logarithmic term is being subtracted from another logarithmic term with the same base number (a above), then the terms can be combined by dividing the arguments (x and y in this case).

Note that the argument which is being taken away (y above) appears on the bottom of the fraction when the two terms are combined.

EXAMPLE

2. Evaluate $\log_4 6 - \log_4 3$.

$$\begin{aligned} &\log_4 6 - \log_4 3 \\ &= \log_4\left(\tfrac{6}{3}\right) \\ &= \log_4 2 \\ &= \tfrac{1}{2} \quad (\text{since } 4^{\frac{1}{2}} = \sqrt{4} = 2). \end{aligned}$$

Rule 3

$$\log_a x^n = n \log_a x \qquad \text{where } a, x > 0.$$

The power of the argument (n above) can come to the front of the term as a multiplier, and vice-versa.

EXAMPLE

3. Express $2\log_7 3$ in the form $\log_7 a$.

$$\begin{aligned} &2\log_7 3 \\ &= \log_7 3^2 \\ &= \log_7 9. \end{aligned}$$

Squash, Split and Fly

You may find the following names are a simpler way to remember the laws of logarithms.

- $\log_a x + \log_a y = \log_a(xy)$ – the arguments are **squashed** together by multiplying.
- $\log_a x - \log_a y = \log_a\left(\tfrac{x}{y}\right)$ – the arguments are **split** into a fraction.
- $\log_a x^n = n \log_a x$ – the power of an argument can **fly** to the front of the log term and vice-versa.

Note

When working with logarithms, you should remember:

$$\log_a 1 = 0 \quad \text{since } a^0 = 1, \qquad\qquad \log_a a = 1 \quad \text{since } a^1 = a.$$

EXAMPLE

4. Evaluate $\log_7 7 + \log_3 3$.

$$\begin{aligned}&\log_7 7 + \log_3 3\\ &= 1 + 1\\ &= 2.\end{aligned}$$

Combining several log terms

When adding and subtracting several log terms in the form $\log_a b$, there is a simple way to combine all the terms in one step.

arguments of **positive** log terms

$$\log_a\left(\frac{\qquad}{\qquad}\right)$$

arguments of **negative** log terms

- Multiply the arguments of the positive log terms in the numerator.
- Multiply the arguments of the negative log terms in the denominator.

EXAMPLES

5. Evaluate $\log_{12} 10 + \log_{12} 6 - \log_{12} 5$.

$$\begin{aligned}&\log_{12} 10 + \log_{12} 6 - \log_{12} 5\\ &= \log_{12}\left(\frac{10 \times 6}{5}\right)\\ &= \log_{12} 12\\ &= 1.\end{aligned}$$

$+\log_{12} \mathbf{10} \qquad +\log_{12} \mathbf{6}$

$$\log_{12}\left(\frac{\qquad}{\qquad}\right)$$

$-\log_{12} \mathbf{5}$

6. Evaluate $\log_6 4 + 2\log_6 3$.

$$\begin{aligned}&\log_6 4 + 2\log_6 3\\ &= \log_6 4 + \log_6 3^2\\ &= \log_6 4 + \log_6 9\\ &= \log_6 (4 \times 9)\\ &= \log_6 36\\ &= 2 \qquad (\text{since } 6^2 = 36).\end{aligned}$$

OR

$$\begin{aligned}&\log_6 4 + 2\log_6 3\\ &= \log_6 2^2 + 2\log_6 3\\ &= 2\log_6 2 + 2\log_6 3\\ &= 2\left(\log_6 2 + \log_6 3\right)\\ &= 2\left(\log_6 (2 \times 3)\right)\\ &= 2\log_6 6\\ &= 2 \qquad (\text{since } \log_6 6 = 1).\end{aligned}$$

4 Exponentials and Logarithms to the Base e

The constant e is an important number in Mathematics, and occurs frequently in models of real-life situations. Its value is roughly 2·718281828 (to 9 d.p.), and is defined as:

$$e = \left(1 + \frac{1}{n}\right)^n \quad \text{as } n \to \infty.$$

If you try very large values of n on your calculator, you will get close to the value of e. Like π, e is an irrational number.

Throughout this section, we will use e in expressions of the form:

- e^x, which is called an exponential to the base e;
- $\log_e x$, which is called a logarithm to the base e. This is also known as the **natural logarithm** of x, and is often written as $\ln x$ (i.e. $\ln x \equiv \log_e x$).

EXAMPLES

1. Calculate the value of $\log_e 8$.

$\log_e 8 = 2{\cdot}08$ (to 2 d.p.). On a calculator: $\boxed{\ln}\ \boxed{8}\ \boxed{=}$

2. Solve $\log_e x = 9$.

$\log_e x = 9$

so $x = e^9$ On a calculator: $\boxed{e^x}\ \boxed{9}\ \boxed{=}$

$x = 8103{\cdot}08$ (to 2 d.p.).

3. Simplify $4\log_e(2e) - 3\log_e(3e)$ expressing your answer in the form $a + \log_e b - \log_e c$ where a, b and c are whole numbers.

$$\begin{aligned}
&4\log_e(2e) - 3\log_e(3e)\\
&= 4\log_e 2 + 4\log_e e - 3\log_e 3 - 3\log_e e\\
&= 4\log_e 2 + 4 - 3\log_e 3 - 3\\
&= 1 + 4\log_e 2 - 3\log_e 3\\
&= 1 + \log_e 2^4 - \log_e 3^3\\
&= 1 + \log_e 16 - \log_e 27.
\end{aligned}$$

OR

$$\begin{aligned}
&4\log_e(2e) - 3\log_e(3e)\\
&= \log_e(2e)^4 - \log_e(3e)^3\\
&= \log_e\left(\frac{(2e)^4}{(3e)^3}\right)\\
&= \log_e\left(\frac{16e^4}{27e^3}\right)\\
&= \log_e\left(\frac{16e}{27}\right)\\
&= \log_e e + \log_e 16 - \log_e 27\\
&= 1 + \log_e 16 - \log_e 27.
\end{aligned}$$

Remember
$(ab)^n = a^n b^n$.

5 Exponential and Logarithmic Equations

Many mathematical models of real-life situations use exponentials and logarithms. It is important to become familiar with using the laws of logarithms to help solve equations.

EXAMPLES

1. Solve $\log_a 13 + \log_a x = \log_a 273$ for $x > 0$.

$$\begin{aligned} \log_a 13 + \log_a x &= \log_a 273 \\ \log_a 13x &= \log_a 273 \\ 13x &= 273 \quad (\text{since } \log_a x = \log_a y \Leftrightarrow x = y) \\ x &= 21. \end{aligned}$$

2. Solve $\log_{11}(4x+3) - \log_{11}(2x-3) = 1$ for $x > \frac{3}{2}$.

$$\begin{aligned} \log_{11}(4x+3) - \log_{11}(2x-3) &= 1 \\ \log_{11}\left(\frac{4x+3}{2x-3}\right) &= 1 \\ \frac{4x+3}{2x-3} &= 11^1 = 11 \quad (\text{since } \log_a x = y \Leftrightarrow x = a^y) \\ 4x+3 &= 11(2x-3) \\ 4x+3 &= 22x-33 \\ 18x &= 36 \\ x &= 2. \end{aligned}$$

3. Solve $\log_a(2p+1) + \log_a(3p-10) = \log_a(11p)$ for $p > 4$.

$$\begin{aligned} \log_a(2p+1) + \log_a(3p-10) &= \log_a(11p) \\ \log_a\left[(2p+1)(3p-10)\right] &= \log_a(11p) \\ (2p+1)(3p-10) &= 11p \\ 6p^2 - 20p + 3p - 10 - 11p &= 0 \\ 6p^2 - 28p - 10 &= 0 \\ (3p+1)(p-5) &= 0 \end{aligned}$$

$$3p+1=0 \quad \text{or} \quad p-5=0$$

$$p = -\tfrac{1}{3} \qquad\qquad p = 5.$$

Since we require $p > 4$, $p = 5$ is the solution.

Dealing with Constants

Sometimes it may help to write constants as logs to solve equations.

EXAMPLE

4. Solve $\log_2 7 = \log_2 x + 3$ for $x > 0$.

Write 3 in logarithmic form:

$$\begin{aligned} 3 &= 3 \times 1 \\ &= 3\log_2 2 \quad (\text{since } \log_2 2 = 1) \\ &= \log_2 2^3 \\ &= \log_2 8. \end{aligned}$$

Use this in the equation:

$$\begin{aligned} \log_2 7 &= \log_2 x + \log_2 8 \\ \log_2 7 &= \log_2 8x \\ 7 &= 8x \\ x &= \tfrac{7}{8}. \end{aligned}$$

OR

$$\begin{aligned} \log_2 7 &= \log_2 x + 3 \\ \log_2 7 - \log_2 x &= 3 \\ \log_2\left(\tfrac{7}{x}\right) &= 3. \end{aligned}$$

Converting from log to exponential form:

$$\begin{aligned} \tfrac{7}{x} &= 2^3 \\ x &= \tfrac{7}{2^3} \\ &= \tfrac{7}{8}. \end{aligned}$$

Solving Equations with unknown Exponents

If an unknown value (e.g. x) is the power of a term (e.g. e^x or 10^x), and its value is to be calculated, then we must take logs on both sides of the equation to allow it to be solved.

The same solution will be reached using any base, but calculators can be used for evaluating logs to the base e and 10.

EXAMPLES

5. Solve $e^x = 7$.

Taking $\log_e$ of both sides:

$$\begin{aligned} \log_e e^x &= \log_e 7 \\ x\log_e e &= \log_e 7 \quad (\log_e e = 1) \\ x &= \log_e 7 \\ x &= 1{\cdot}946 \quad (\text{to 3 d.p.}). \end{aligned}$$

OR Taking $\log_{10}$ of both sides:

$$\begin{aligned} \log_{10} e^x &= \log_{10} 7 \\ x\log_{10} e &= \log_{10} 7 \\ x &= \frac{\log_{10} 7}{\log_{10} e} \\ x &= 1{\cdot}946 \quad (\text{to 3 d.p.}). \end{aligned}$$

6. Solve $5^{3x+1} = 40$.

$$\begin{aligned}
\log_e 5^{3x+1} &= \log_e 40 \\
(3x+1)\log_e 5 &= \log_e 40 \\
3x+1 &= \frac{\log_e 40}{\log_e 5} \\
3x+1 &= 2{\cdot}2920 \\
3x &= 1{\cdot}2920 \\
x &= 0{\cdot}431 \quad \text{(to 3 d.p.)}.
\end{aligned}$$

Note
$\log_{10}$ could have been used instead of $\log_e$.

Exponential Growth and Decay

Recall from Section 1 that exponential functions are sometimes known as growth or decay functions. These often occur in models of real-life situations.

For instance, radioactive decay can be modelled using an exponential function. An important measurement is the **half-life** of a radioactive substance, which is the time taken for the mass of the radioactive substance to halve.

EXAMPLE

7. The mass G grams of a radioactive sample after time t years is given by the formula $G = 100e^{-3t}$.

(a) What is the initial mass of radioactive substance in the sample?

(b) Find the half-life of the radioactive substance.

(a) The initial mass was present when $t = 0$:

$$\begin{aligned}
G &= 100e^{-3\times 0} \\
&= 100e^{0} \\
&= 100.
\end{aligned}$$

So the initial mass was 100 grams.

(b) The half-life is the time t at which $G = 50$, so

$$\begin{aligned}
100e^{-3t} &= 50 \\
e^{-3t} &= \tfrac{1}{2} \\
-3t &= \log_e\left(\tfrac{1}{2}\right) \qquad \text{(converting to log form)} \\
t &= 0{\cdot}231 \text{ (to 3 d.p.)}.
\end{aligned}$$

So the half-life is $0{\cdot}231$ years, roughly $0{\cdot}231 \times 356 = 84{\cdot}315$ days.

8. The world population, in billions, t years after 1950 is given by
$P = 2{\cdot}54e^{0{\cdot}0178t}$.

(a) What was the world population in 1950?

(b) Find, to the nearest year, the time taken for the world population to double.

(a) For 1950, $t = 0$:

$$\begin{aligned} P &= 2{\cdot}54e^{0{\cdot}0178\times 0} \\ &= 2{\cdot}54e^{0} \\ &= 2{\cdot}54. \end{aligned}$$

So the world population in 1950 was $2{\cdot}54$ billion.

(b) For the population to double:

$$\begin{aligned} 2{\cdot}54e^{0{\cdot}0178t} &= 2\times 2{\cdot}54 \\ e^{0{\cdot}0178t} &= 2 \\ 0{\cdot}0178t &= \log_e 2 \qquad \text{(converting to log form)} \\ t &= 38{\cdot}94 \text{ (to 2 d.p.).} \end{aligned}$$

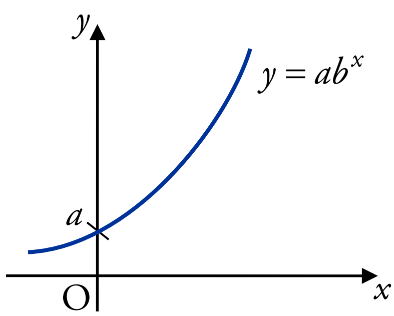
So the population doubled after 39 years (to the nearest year).

6 Graphing with Logarithmic Axes EF

It is common in applications to find an exponential relationship between variables; for instance, the relationship between the world population and time in the example above. Given some data (e.g. from an experiment) we would like to find an explicit equation for the relationship.

Relationships of the form $y = ab^x$

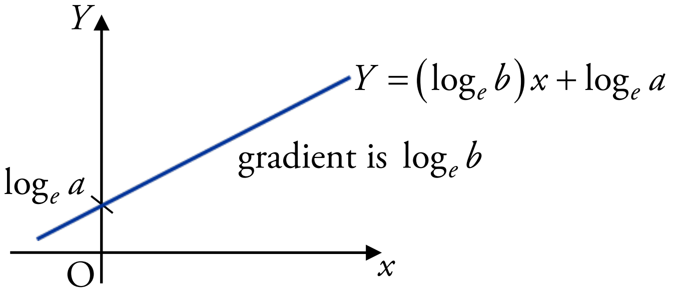
Suppose we have an exponential graph $y = ab^x$, where $a, b > 0$.



Taking logarithms we find that

$$\begin{aligned}\log_e y &= \log_e\left(ab^x\right)\\ &= \log_e a + \log_e b^x\\ &= \log_e a + x\log_e b.\end{aligned}$$

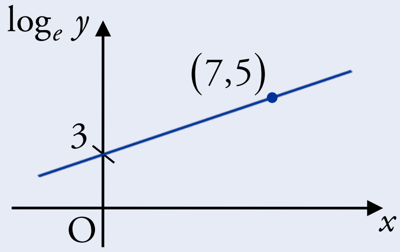
We can scale the y-axis so that $Y = \log_e y$; the Y-axis is called a **logarithmic axis**. Now our relationship is of the form $Y = \left(\log_e b\right)x + \log_e a$, which is a straight line in the (x, Y)-plane.



Since this is just a straight line, we can use known points to find the gradient $\log_e b$ and the Y-axis intercept $\log_e a$. From these we can easily find the values of a and b, and hence specify the equation $y = ab^x$.

EXAMPLES

1. The relationship between two variables, x and y, is of the form $y = ab^x$, where a and b are constants. An experiment to test this relationship produced the data shown in the graph, where $\log_e y$ is plotted against x.



Find the values of a and b.

We ned to obtain a straight line equation:

$$\begin{aligned} y &= ab^x\\ \log_e y &= \log_e ab^x \qquad \text{(taking logs of both sides)}\\ \log_e y &= \log_e a + \log_e b^x\\ \log_e y &= \log_e a + x\log_e b\\ Y &= \left(\log_e b\right)x + \log_e a.\end{aligned}$$

From the graph, the Y-axis intercept is $\log_e a = 3$; so $a = e^3$.

Using the gradient formula:

$$\begin{aligned}\log_e b &= \frac{5-3}{7-0}\\ &= \tfrac{2}{7}\\ b &= e^{\frac{2}{7}}.\end{aligned}$$

2. The results from an experiment were noted as follows:

x	1·30	2·00	2·30	2·80
$\log_e y$	2·04	2·56	2·78	3·14

The relationship between these data can be written in the form $y = ab^x$.

Find the values of a and b, and state the formula for y in terms of x.

We need to obtain a straight line equation:

$$\begin{aligned}y &= ab^x\\ \log_e y &= \log_e ab^x \qquad \text{(taking logs of both sides)}\\ \log_e y &= \log_e a + \log_e b^x\\ \log_e y &= \log_e a + x\log_e b\\ \log_e y &= (\log_e b)x + \log_e a.\end{aligned}$$

We can find the gradient $\log_e b$ (and hence b), using two points on the line: using $(1{\cdot}30, 2{\cdot}04)$ and $(2{\cdot}80, 3{\cdot}14)$,

$$\log_e b = \frac{3{\cdot}14 - 2{\cdot}04}{2{\cdot}80 - 1{\cdot}30} = 0{\cdot}73 \text{ (to 2 d.p.)}.$$

So $b = e^{0.73} = 2{\cdot}08$ (to 2 d.p.).

Note that $\log_e y = 0{\cdot}73x + \log_e a$. We can work out $\log_e a$ (and hence a) by substituting a point into this equation: using $(1{\cdot}30, 2{\cdot}04)$,

$$\begin{aligned}2{\cdot}04 &= 0{\cdot}73 \times 1{\cdot}30 + \log_e a\\ \log_e a &= 2{\cdot}04 - 0{\cdot}73 \times 1{\cdot}30\\ &= 1{\cdot}09 \text{ (to 2 d.p.)}\\ a &= e^{1{\cdot}09}\\ &= 2{\cdot}97 \text{ (to 2 d.p.)}.\end{aligned}$$

Therefore $y = 2{\cdot}97 \times 2{\cdot}08^x$.

Note
Depending on the points used, slightly different values for *a* and *b* may be obtained.

Equations in the form $y = ax^b$

Another common relationship is $y = ax^b$, where $a > 0$. In this case, the relationship can be represented by a straight line if we change *both* axes to logarithmic ones.

EXAMPLE

3. The results from an experiment were noted as follows:

$\log_{10} x$	1·70	2·29	2·70	2·85
$\log_{10} y$	1·33	1·67	1·92	2·01

The relationship between these data can be written in the form $y = ax^b$.

Find the values of a and b, and state the formula for y in terms of x.

We need to obtain a straight line equation:

$$\begin{aligned} y &= ax^b \\ \log_{10} y &= \log_{10} ax^b \quad \text{(taking logs of both sides)} \\ \log_{10} y &= \log_{10} a + \log_{10} x^b \\ \log_{10} y &= \log_{10} a + b\log_{10} x \\ Y &= bX + \log_{10} a. \end{aligned}$$

We can find the gradient b, using two points on the line: using $(1{\cdot}70, 1{\cdot}33)$ and $(2{\cdot}85, 2{\cdot}01)$,

$$b = \frac{2{\cdot}01 - 1{\cdot}33}{2{\cdot}85 - 1{\cdot}70} = 0{\cdot}59 \quad \text{(to 2 d.p.)}$$

So $\log_{10} y = 0{\cdot}59\log_{10} x + \log_{10} a$.

Now we can work out a by substituting a point into this equation: using $(1{\cdot}70, 1{\cdot}33)$,

$$\begin{aligned} 1{\cdot}33 &= 0{\cdot}59 \times 1{\cdot}70 + \log_{10} a \\ \log_{10} a &= 1{\cdot}33 - 0{\cdot}59 \times 1{\cdot}70 \\ &= 0{\cdot}33 \\ a &= 10^{0{\cdot}33} \\ &= 2{\cdot}14 \quad \text{(to 2 d.p.).} \end{aligned}$$

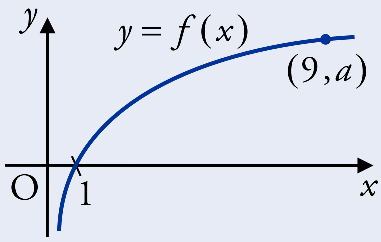
Therefore $y = 2{\cdot}14x^{0{\cdot}59}$.

7 Graph Transformations

Graph transformations were covered in the notes on Functions and Graphs, but we can now look in more detail at applying transformations to graphs of exponential and logarithmic functions.

EXAMPLES

1. Shown below is the graph of $y = f(x)$ where $f(x) = \log_3 x$.



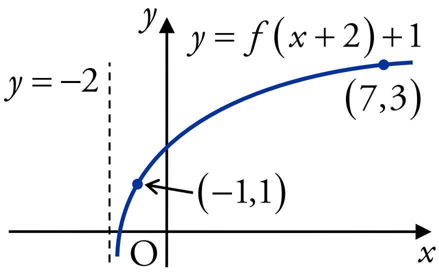
(a) State the value of a.

(b) Sketch the graph of $y = f(x+2)+1$.

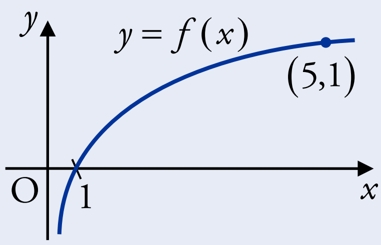
(a) $a = \log_3 9$

$= 2 \qquad (\text{since } 3^2 = 9)$.

(b) The graph shifts two units to the left, and one unit upwards:



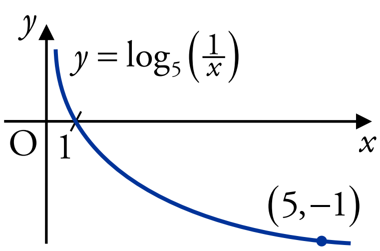
2. Shown below is part of the graph of $y = \log_5 x$.



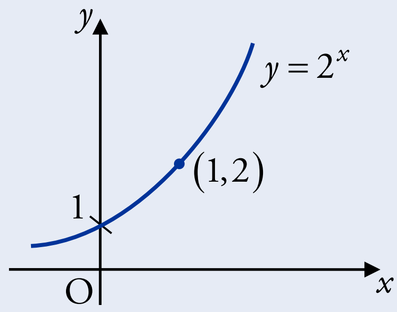
Sketch the graph of $y = \log_5\left(\frac{1}{x}\right)$.

$$
\begin{aligned}
y &= \log_5\left(\frac{1}{x}\right) \\
&= \log_5 x^{-1} \\
&= -\log_5 x
\end{aligned}
$$

So reflect in the x-axis.



3. The diagram shows the graph of $y = 2^x$.

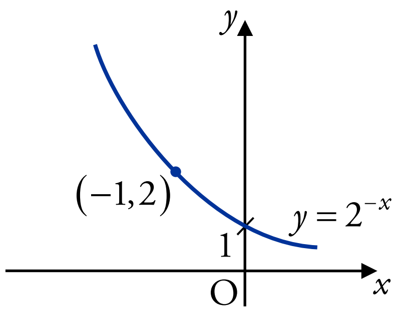


On separate diagrams, sketch the graphs of:

(a) $y = 2^{-x}$;

(b) $y = 2^{2-x}$.

(a) Reflect in the y-axis:

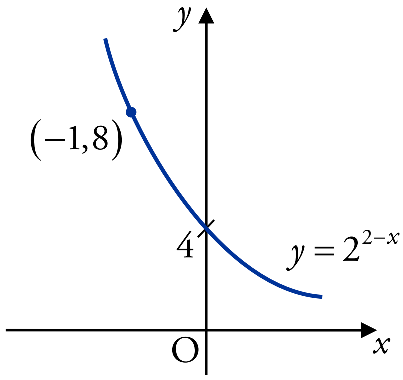


(b) $y = 2^{2-x}$

$= 2^2 2^{-x}$

$= 4 \times 2^{-x}$.

So scale the graph from (a) by 4 in the y-direction:



Exercise 1 : Exponentials

1 a) Complete this table of values for $f(x) = e^x$.
(Use the e^x key on your calculator. Round each value correct to one decimal place.)

x	-3	-2	-1	0	1	2	3
$f(x)$							

b) Sketch the graph of $f(x) = e^x$.

2 Evaluate, correct to three significant figures:

a) e^x when $x = 0{\cdot}7$

b) e^t when $t = 1{\cdot}7$

c) e^{-n} when $n = 2{\cdot}5$

d) e^{2a} when $a = 4$

e) $e^{-0{\cdot}012k}$ when $k = 100$

f) $50e^{-3c}$ when $c = 2$

Exercise 2 : Exponential Growth and Decay

1 The population of a town is 12 000. It is estimated that the population of the town will increase at the rate of 15% per annum.

a) Write down a formula for P_n, the population of the town after n years.
b) Calculate the estimated population after 6 years.

2 The value of a car when new is £25 000. It is estimated that the value of the car will decrease at the rate of 12% per annum.

a) Write down a formula for V_n, the value of the car after n years.
b) Calculate the estimated value of the car after 10 years.

3 The population of the world in 2000 was 6·08 billion. It is estimated that the annual rate of increase is 1·25%.

a) Write down a formula for P_n, the population of the world n years after 2000.
b) Calculate the estimated population in 2016.

4 A 500 millilitre puddle of water is evaporating at a rate of 15% per hour.

a) Write down a formula for W_h, the amount of water in the puddle after h hours.
b) How much water is left in the puddle after 4 hours?

5 Jack borrows £800 from a loan company. He must repay the loan within a year. He can make regular payments throughout the year or a single payment at the end of the year. Interest is charged on any unpaid balance at a rate of 2% per month. Jack chooses to make a single payment at the end of the year. How much will the payment be?

6 A diamond ring was bought a number of years ago for £500. The value of the ring increases by 8% per year. How many years does it take for the ring to double in value?

7 A hospital patient is given a 400 milligram dose of a drug. Each hour, the amount of drug in the patient's body decreases by 30%. When there is less than 100 milligrams of the drug left in the patient's body, it is safe for the patient to be given another dose. How long will it be before the patient can be given another dose?

8 The number of bacteria, N, in a liquid culture is given by the formula $N = 500e^{13t}$, where t is the time in hours.

a) How many bacteria are there at time zero?
b) How many bacteria are there after three hours?

9 The mass, m milligram of a radioactive substance is given by the formula $m = 200e^{-0{\cdot}1t}$, where t represents the time in years.

a) What is the mass of the substance at time zero?
b) What is the mass of the substance after four years?

Exercise 3 : Logarithms

1 Express in logarithmic form: [e.g. $5^2 = 25 \Rightarrow \log_5 25 = 2$]

a) $2^3 = 8$ b) $3^2 = 9$ c) $4^3 = 64$ d) $100 = 10^2$

e) $3^{-2} = \frac{1}{9}$ f) $10^{-3} = \frac{1}{1000}$ g) $3^0 = 1$ h) $4^1 = 4$.

2 Express in index form: [e.g. $\log_4 16 = 2 \Rightarrow 4^2 = 16$]

a) $\log_5 25 = 2$ b) $\log_6 36 = 2$ c) $\log_4 64 = 3$ d) $\log_2 \frac{1}{8} = -3$

e) $\log_2 \sqrt{2} = \frac{1}{2}$ f) $\log_5 \sqrt[3]{5} = \frac{1}{3}$ g) $\log_2 1 = 0$ h) $\log_5 5 = 1$.

3 Express in logarithmic form:

a) $p^2 = q$ b) $r = s^3$ c) $a = b^c$ d) $k^t = v$.

4 Express in indicial form:

a) $\log_3 M = N$ b) $t = \log_4 u$ c) $k = \log_p m$ d) $\log_b a = c$.

5 Solve for x:

a) $\log_3 x = 2$ b) $\log_4 x = 3$ c) $\log_{\sqrt{2}} x = 4$ d) $\log_{\sqrt{3}} x = 8$

e) $4 = \log_{\frac{1}{2}} x$ f) $2 = \log_{\frac{1}{3}} x$ g) $-1 = \log_2 x$ h) $-3 = \log_3 x$

i) $\log_x 8 = 3$ j) $\log_x \frac{1}{16} = -2$ k) $\log_x \frac{1}{27} = -3$ l) $\log_x 4 = 4$.

6 Evaluate: [e.g. $\log_3 9 = x \Rightarrow 3^x = 9 = 3^2 \Rightarrow x = 2$]

a) $\log_2 8$ b) $\log_3 81$ c) $\log_9 81$ d) $\log_2 \frac{1}{4}$

e) $\log_{\frac{1}{2}} 4$ f) $\log_{\frac{1}{3}} 27$ g) $\log_9 27$ h) $\log_8 \frac{1}{16}$.

Exercise 4 : Laws of Logarithms

1 Express as the logarithm of a single number:

a) $\log 3 + \log 5$ b) $\log 10 - \log 2$ c) $5 \log 2$

d) $\log 14 + \log 2$ e) $\frac{1}{2} \log 16$ f) $\log 27 - \log 18$

g) $-3 \log 2$ h) $\log 1 + \log 2$.

2 Express as the logarithm of a single number:

a) $2 \log 2 + \log 3$ b) $\log 128 - 4 \log 2$ c) $4 \log 2 - 2 \log 4$

d) $3 \log 2 - 2 \log 3$ e) $2 \log\sqrt{5} - \frac{1}{2} \log 5$ f) $\log 6 - 2 \log 2 + \log 8$.

3 Express as the logarithm of a single number: [e.g. $1 + \log_3 5 = \log_3 3 + \log_3 5 = \log_3 15$]

a) $1 + \log_2 3$ b) $1 - \log_3 5$ c) $\log_2 6 - 2$

d) $2\log_3 5 - 1$ e) $2 + 3\log_3 3$ f) $3 - 2\log_2 2$

g) $1 - \log_{10} 2$ h) $2 + \log_{10} 5$ i) $1 + \log_{10} 3 - \log_{10} 2$.

4 Express as the logarithm of a single number:

a) $\log p + \log q$ b) $2\log r - \log s$ c) $\log t + \frac{1}{2}\log u$

d) $\log mn - 2\log n$ e) $\log xy - \frac{1}{2}\log x$ f) $\log r + 2\log s - 3\log t$.

5 Express as the logarithm of a single number:

a) $1 + \log_{10} a$ b) $2 - 3\log_{10} b$ c) $\log_{10} a + \frac{1}{2}\log_{10} b - 1$.

6 Solve for x:

a) $\log x + \log 2 = \log 10$ b) $\log x - \log 4 = \log 25$

c) $\log x + 2\log 3 = \log 54$ d) $3\log 6 - \log x = \log 18$

e) $\log(x+1) + \log(x-1) = \log 8$ f) $\log x^2 + \log 0{\cdot}25 = \log 4$.

Exercise 5 : Further manipulation with Logarithms

1 Express each of the following in terms of log 2 and log 3:

a) $\log 8$ b) $\log 9$ c) $\log 12$ d) $\log 18$

e) $\log \frac{1}{2}$ f) $\log \frac{1}{9}$ g) $\log \frac{1}{6}$ h) $\log 0{\cdot}75$.

2 a) Express $\log_{10} 5$ in terms of $\log_{10} 2$.

b) Hence, or otherwise, express each of the following in terms of $\log_{10} 2$ and $\log_{10} 3$:

(i) $\log_{10} \frac{1}{5}$ (ii) $\log_{10} 25$ (iii) $\log_{10} 15$ (iv) $\log_{10} 20$

(v) $\log_{10} 75$ (vi) $\log_{10} \frac{1}{125}$ (vii) $\log_{10} \frac{5}{16}$ (viii) $\log_{10} 0{\cdot}6$.

3B If $p = \log_a x$ and $q = \log_{x^2} a$, find the connection between p and q.

4B If $a^p = x$, $a^q = y$ and $a^r = z$, calculate the value of $\log_a \frac{xy}{z^2}$.

5B If $a = \log x^2 y$ and $b = \log \frac{x}{y^2}$, express $\log x$ and $\log y$ in terms of a and b.

6B If $3\log_2 y = 2\log_2(x-3) + 5$, show that $y^3 = 32(x-3)^2$.

7 Solve for x, y, z: a) $5^x = 9$ b) $3^y = 7$ c) $2^z = \frac{5}{4}$.

8 Solve for p, q, r: a) $p = \log_3 17$ b) $q = \log_7 25$ c) $\log_3 0{\cdot}3 = r$.

Exercise 6 : Logs to base 10 and Natural Logarithms

1 Use your calculator to write down, correct to three decimal places, the value of:

a) $\log_{10} 2$ **b)** $\log_{10} 3$ **c)** $\log_{10} 5$

d) $\log_{10} 900$ **e)** $\log_{10} \sqrt{10}$ **f)** $\log_{10} 0{\cdot}8$

2 Use your calculator to write down, correct to three decimal places, the value of:

a) $\ln 2$ **b)** $\ln 3$ **c)** $\ln 5$

d) $\ln 900$ **e)** $\ln \sqrt{10}$ **f)** $\ln 0{\cdot}8$

3 a) Complete this table of values for $\log_{10} ab$, $\log_{10} a \times \log_{10} b$ and $\log_{10} a + \log_{10} b$.
Give each value correct to three decimal places.

	$\log_{10} ab$	$\log_{10} a \times \log_{10} b$	$\log_{10} a + \log_{10} b$
$a = 2$, $b = 3$			
$a = 4$, $b = 7$			
$a = 8$, $b = 6$			

b) Make a conjecture based on the values in your table: $\log_{10} ab = \ldots$

4 a) Complete this table of values for $\ln ab$, $\ln a \times \ln b$, $\ln a + \ln b$.
Give each value correct to three decimal places.

	$\ln ab$	$\ln a \times \ln b$	$\ln a + \ln b$
$a = 2$, $b = 3$			
$a = 4$, $b = 7$			
$a = 8$, $b = 6$			

b) Make a conjecture based on the values in your table: $\ln ab = \ldots$

5 a) Complete this table of values for $\log_{10} \frac{a}{b}$, $\frac{\log_{10} a}{\log_{10} b}$ and $\log_{10} a - \log_{10} b$.
Give each value correct to three decimal places.

	$\log_{10} \frac{a}{b}$	$\frac{\log_{10} a}{\log_{10} b}$	$\log_{10} a - \log_{10} b$
$a = 12$, $b = 4$			
$a = 26$, $b = 2$			
$a = 15$, $b = 3$			

b) Make a conjecture based on the values in your table: $\log_{10} \frac{a}{b} = \ldots$

6 a) Complete this table of values for $\ln\frac{a}{b}$, $\frac{\ln a}{\ln b}$ and $\ln a - \ln b$.

Give each value correct to three decimal places.

	$\ln\frac{a}{b}$	$\frac{\ln a}{\ln b}$	$\ln a - \ln b$
$a = 12, b = 4$			
$a = 26, b = 2$			
$a = 15, b = 3$			

b) Make a conjecture based on the values in your table: $\ln\frac{a}{b} = \ldots$

7 a) Complete this table of values for $\log_{10} a^b$, $(\log_{10} a)^b$ and $b \log_{10} a$.

Give each value correct to three decimal places.

	$\log_{10} a^b$	$(\log_{10} a)^b$	$b \log_{10} a$
$a = 5, b = 2$			
$a = 4, b = 3$			
$a = 2, b = 6$			

b) Make a conjecture based on the values in your table: $\log_{10} a^b = \ldots$

8 a) Complete this table of values for $\ln a^b$, $(\ln a)^b$ and $b \ln a$.

Give each value correct to three decimal places.

	$\ln a^b$	$(\ln a)^b$	$b \ln a$
$a = 5, b = 2$			
$a = 4, b = 3$			
$a = 2, b = 6$			

b) Make a conjecture based on the values in your table: $\ln a^b = \ldots$

Exercise 7 : Logarithmic Equations

1 Solve for $x > 0$:

a) $\log_a 4 + \log_a 3x = \log_a 60$ **b)** $2\log_a 3 + \log_a x = \log_a 54$

c) $\log_4 x + \log_4(x+1) = \log_4 2$ **d)** $\log_b(2x+1) - \log_b(x-2) = \log_b 3$

2 Solve for $x > 0$: HPQ

a) $\log_4 x = \log_4 96 - 2$ **b)** $3 + \log_2 x = \log_2 7$

c) $\log_3 x = 3 - \log_3 5$ **d)** $\log_5 x - 3\log_5 2 = 1$

e) $\log_6(x+3) + \log_6(x-2) = 1$ **f)** $\log_3 3x - \log_3(x-2) = 2$

g) $\log_{11}(4x+3) - \log_{11}(2x-3) = 1$ **h)** $\frac{1}{2}\log_5 16 + 2\log_5 x = 2$

3 Solve for $x > 0$:

a) $\log_x 80 - \log_x 5 = 2$ **b)** $2\log_x 5 + \log_x 40 = 3$

c) $\frac{1}{2}\log_x 16 + 3\log_x 2 = 5$ **d)** $2\log_x 6 - \frac{2}{3}\log_x 8 = 2$

4 Solve the following equations, giving your answers correct to two decimal places:

a) $6^x = 40$ **b)** $2^y = 35$

c) $3^a = 5$ **d)** $10^b = 5$

e) $4^{2x-1} = 9$ **f)** $3^{2m+3} = 7$

5 Solve the following equations, giving your answers correct to two decimal places:

a) $p = \log_3 21$ **b)** $q = \log_4 19$

6 Find the coordinates of the points where the following curves cross the x-axis: ACE

a) $y = \log_4 x - 2$ **b)** $y = \log_3(x-4) - 1$

c) $y = \log_2(x+1) + 1$

Exercise 8 : Exponential Growth and Decay

1 The intensity, I_0 units, of a source of light is diminished to I_d units on passing through d metres of fog, according to the law $I_d = I_0e^{-0\cdot15d}$.
 a) Calculate the intensity of illumination 20 metres away from a 250 unit light source through this fog.
 b) At what distance from this source will the intensity of illumination be 56 units?

2 The current, I_d amps, in a telephone wire d kilometres along the wire from where the initial current strength is I_0 amps, reduces according to the law $I_d = I_0\ e^{-0\cdot13d}$.
 a) Calculate the current at the end of a 10 km wire with inital current 10 amps.
 b) At what distance along this wire has the current fallen to 6 amps?

3 A number N_0 of radioactive nuclei decay to N_t after t years according to the law $N_t = N_0\ e^{-0\cdot05t}$.
 a) Find the number remaining after 50 years if the original number N_0 was 500.
 b) The half-life of a radioactive sample is defined as the time taken for the activity to be reduced by half. Calculate the half-life for this sample.

4 A tractor tyre has a slow puncture which causes the pressure within it to drop. The pressure, P_t units, t hours after inflation to P_0 units is governed by the relationship $P_t = P_0\ e^{-kt}$.
 a) The tyre is inflated to a pressure of 50 units. Twenty-four hours later the pressure has dropped to 10 units. Calculate the value of k to three decimal places.
 b) The tyre manufacturer advises that serious damage to the tyre will result if it is used when the pressure drops below 30 units.
 If the farmer inflates the tyre to 50 units and uses the tractor for four hours, will he have caused any serious damage to this tyre?

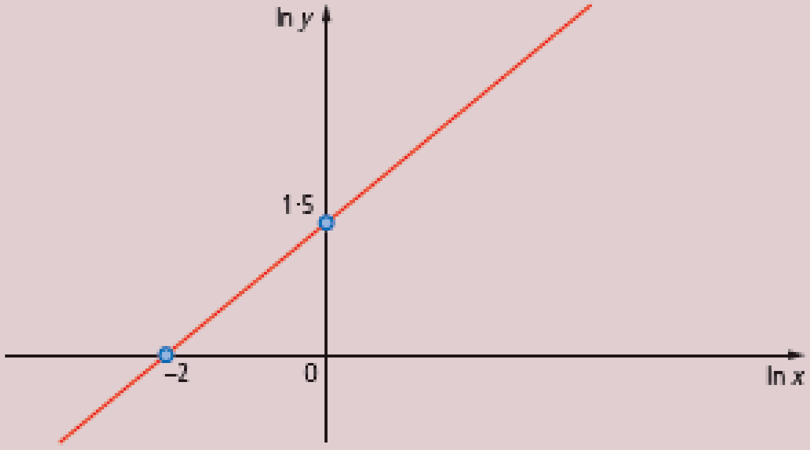
5 A mug of tea cools from an initial temperature of T_0°C to T_t°C in t minutes according to the law $T_t = T_0\ e^{-kt}$.
 a) A mug of tea cooled from boiling (100°C) to 75° C in 7 minutes.
 Calculate the value of k to three significant figures.
 b) How much longer did it take for this mug of tea to cool to the room temperature of 20°C?

6 The intensity, I_0 units, of a source of light is diminished to I_t units on passing through a filter of thickness t centimetres, according to the law $I_t = I_0e^{-kt}$.
 a) If the intensity is reduced by one quarter on passing through a filter which is 4 cm thick, calculate the value of k to three significant figures.
 b) By what percentage would the intensity be reduced if the light passed through a filter of the same material which was 10 cm thick?

7 The atmospheric pressure, P_h millimetres of mercury, at a height of h kilometres above sea level, on a day when the atmospheric pressure at sea level is P_0 mm of mercury, is given by $P_h = P_0e^{-kh}$. If, when the pressure at sea level is 760 mm, the pressure at a height of 1 km is 670 mm, calculate the pressure at a height of 2 km.

Exercise 9 :

1 The results of an experiment led to the graph shown below.

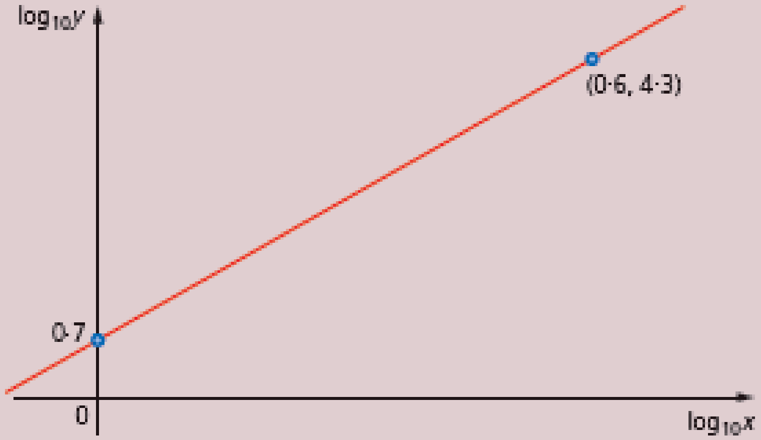


a) Write down the equation of the line in terms of ln y and ln x.
b) Show that x and y satisfy a relationship of the form $y = ax^b$ and find the values of a and b.

2 Two variables x and y satisfy the relationship $y = 5x^3$.
When $\log_{10} y$ is plotted against $\log_{10} x$ a straight line graph is obtained.
Find the coordinates of the point where this line meets the y-axis.

3 Show that each graph represents a function of the form $y = ax^b$ and determine the values of a and b.

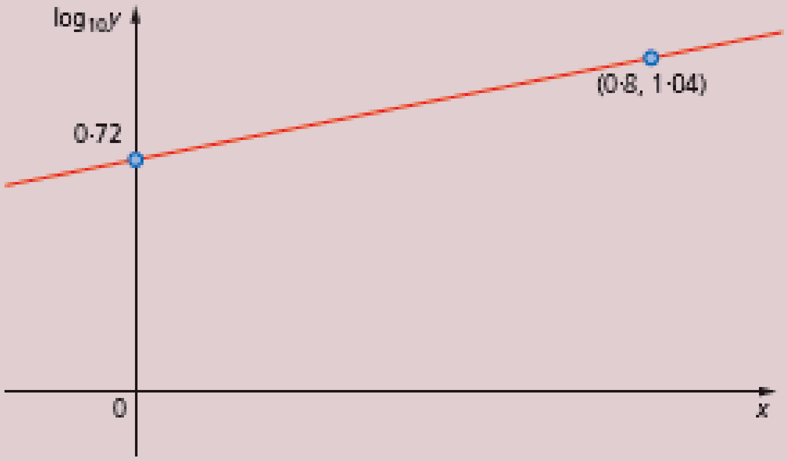
a)



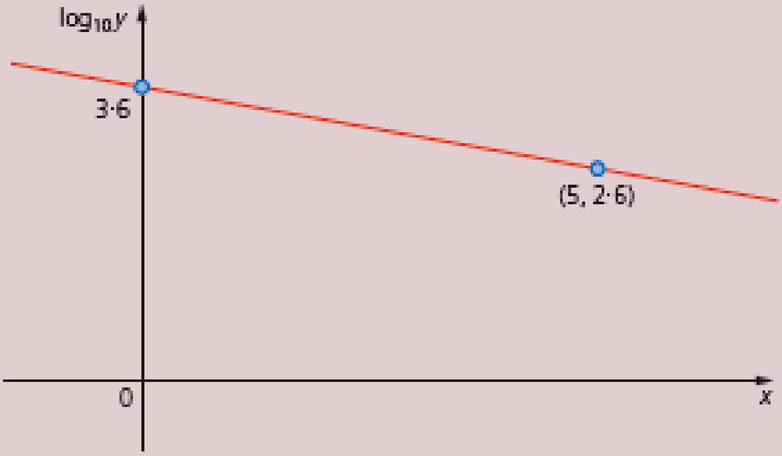
b)

4 Show that each graph represents a function of the form $y = ab^x$ and find the values of a and b.

a)



b)



5 This table of data was obtained from an experiment.

x	1·1	2·3	3·1	4·4
y	3·7	18·7	36·2	78·1

a) Show that the variables x and y are related by the formula $y = ax^b$.
b) Find the values of a and b.

6 Repeat question 5 for the data shown in the table below.

x	10	20	30	40
y	1·1	1·5	1·7	2·0

7 This table of data was obtained from an experiment.

x	1	2	3	4
y	8·5	42·5	212·5	1062·5

a) Show that the variables x and y are related by the formula $y = ab^x$.
b) Find the values of a and b.

Exercise 10 :

1 Given $\boldsymbol{X} = \log_{10} x$ and $\boldsymbol{Y} = \log_{10} y$, convert each of these to the form $\boldsymbol{Y} = a\boldsymbol{X} + b$:
a) $y = x^2$ b) $y = 4x^3$ c) $y = 5x$.

2 Given $\boldsymbol{X} = \log_e x$ and $\boldsymbol{Y} = \log_e y$, convert each of these to the form $\boldsymbol{Y} = a\boldsymbol{X} + b$:
a) $y = 3x^4$ b) $y = 10x^{-1}$ c) $y = 0{\cdot}25x$.

3 Given $\boldsymbol{Y} = \log_{10} y$, convert each of the following to the form $\boldsymbol{Y} = a + bx$:
a) $y = 2 \times 10^x$ b) $y = 7 \times 3^x$ c) $y = 5 \times 2^x$.

4 Given $\boldsymbol{Y} = \log_e y$, convert each of following to the form $\boldsymbol{Y} = a + bx$:
a) $y = 5e^x$ b) $y = 2e^{-3x}$ c) $y = 8 \times 5^x$.

5 Express $P = 4Q + 0{\cdot}48$ in the form $p = aq^n$ where $P = \log_{10} p$ and $Q = \log_{10} q$.

6 Express $M = 3{\cdot}6V + 0{\cdot}43$ in the form $m = av^n$ where $M = \log_{10} m$ and $V = \log_{10} v$.

7 Express $S = 2{\cdot}1T - 0{\cdot}7$ in the form $s = at^n$ where $S = \log_{10} s$ and $T = \log_{10} t$.

8 Express $Y = 1{\cdot}39 - 2X$ in the form $y = ax^n$ where $Y = \log_e y$ and $X = \log_e x$.

9 Express $W = 0{\cdot}85 - 3X$ in the form $w = ax^n$ where $W = \log_{10} w$ and $X = \log_{10} x$.

10 Express $P = 1{\cdot}7Z + 0{\cdot}69$ in the form $p = az^n$ where $P = \log_e p$ and $Z = \log_e z$.

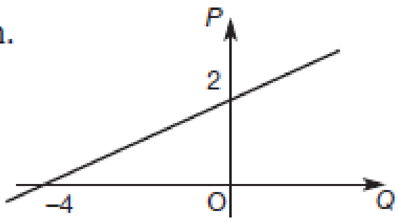
11 Express $S = 4T + 0{\cdot}7$ in the form $s = at^n$ where $S = \log_{10} s$ and $T = \log_{10} t$.

12 Express $R = -0{\cdot}9 - 2V$ in the form $r = av^n$ where $R = \log_e r$ and $V = \log_e v$.

13 Express $F = 0{\cdot}5 + 0{\cdot}6t$ in the form $f = ab^t$ where $F = \log_{10} f$.

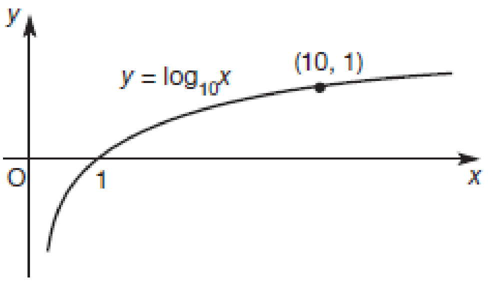
14 Express $G = 1{\cdot}61 + 3r$ in the form $g = ab^r$ where $G = \log_e g$.

15B The results of an experiment gave rise to the graph shown.
a) Find the equation of the line (in terms of P and Q).
b) Given that $P = \log_{10} p$ and $Q = \log_{10} q$, show that p and q satisfy a relationship of the form $p = aq^b$, stating the values of a and b.



Exercise 11 : Graph Transformations

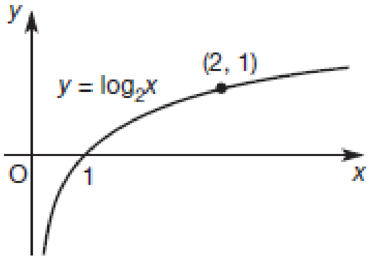
1 The graph of $y = \log_{10} x$ is shown.
Sketch the graph of:
a) $y = \log_{10}(x + 2)$ b) $y = \log_{10}(x - 2)$
c) $y = \log_{10}(-x)$ d) $y = 1 - \log_{10} x$.
[Remember to indicate all asymptotes and intersections with axes.]



2B Sketch the graph of each of the following functions by first applying the laws of logs to express the function in a more suitable form for graphing:

a) $\log_{10} x^2\ (x > 0)$ b) $\log_{10}\left(\frac{1}{x}\right)$ c) $\log_{10} 10x$ d) $\log_{10}\left(\frac{x}{100}\right)$.

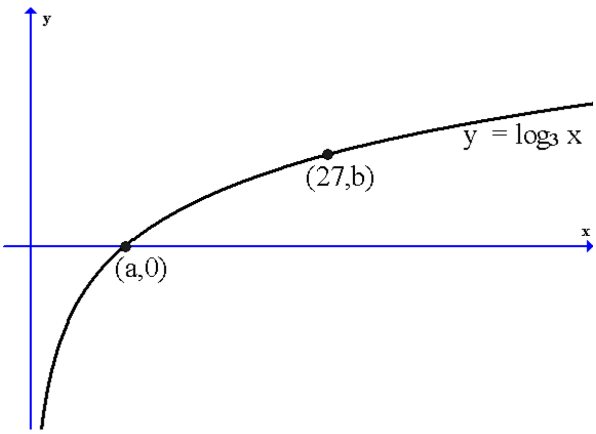
3B The graph of $y = \log_2 x$ is shown.
As in question 2B sketch the graph of:
a) $y = \log_2 2x$ b) $y = \log_2 x^3$
c) $y = 1 + \log_2(x + 1)$ d) $y = 1 - \log_2 \sqrt{x}$.



Exercise 12

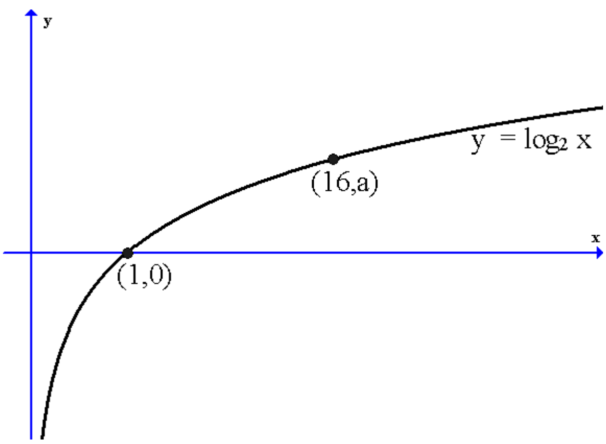
1. The diagram shows part of the graph of $y = \log_3 x$.

   (a) Find the values of a and b.
   (b) Sketch the graph of $y = \log_3(x + 1) - 3$.



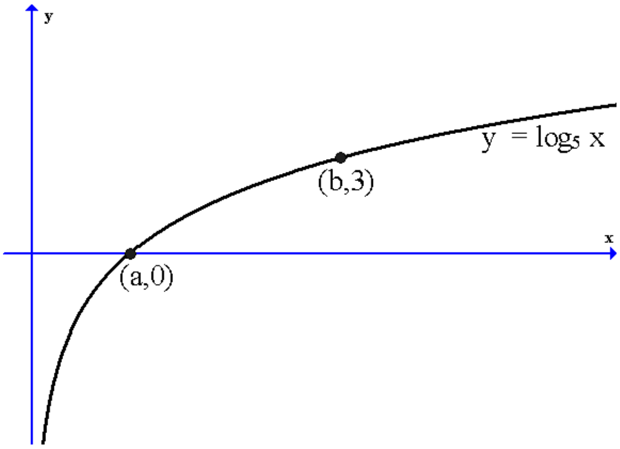
2. The diagram shows part of the graph of $y = \log_2 x$.

   (a) Find the value of a.
   (b) Sketch the graph of $y = \log_2 x - 4$.
   (c) Sketch the graph of $y = \log_2 8x$.



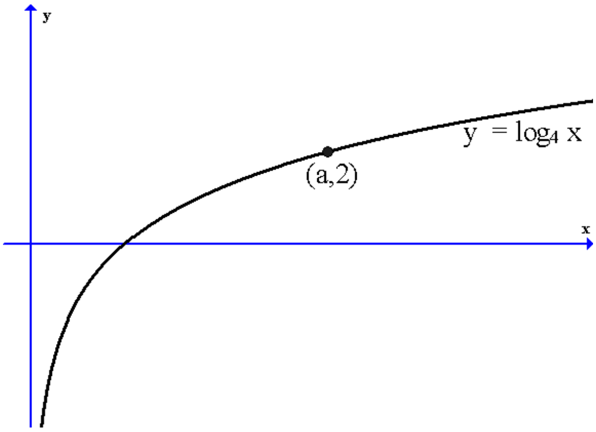
3. The diagram shows part of the graph of $y = \log_5 x$.

   (a) Find a and b.
   (b) Sketch the graph of $y = \log_5 5x$.
   (c) Sketch the graph of $y = \log_5 x^2$
   (d) Sketch the graph of $y = \log_5 \frac{1}{x}$.



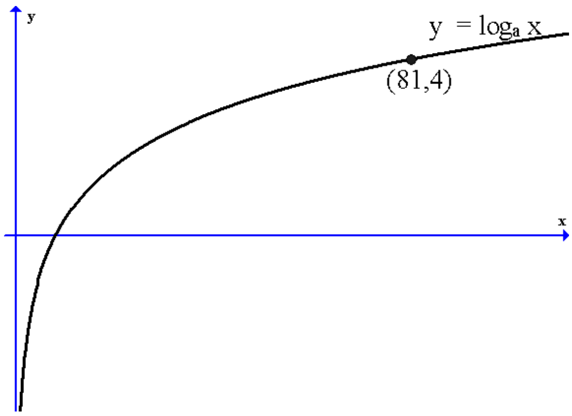
4. The diagram shows part of the graph of $y = \log_4 x$.

   (a) Find a.
   (b) Sketch the graph of $y = \log_4 4x$.
   (c) Sketch the graph of $y = \log_4 x^3$



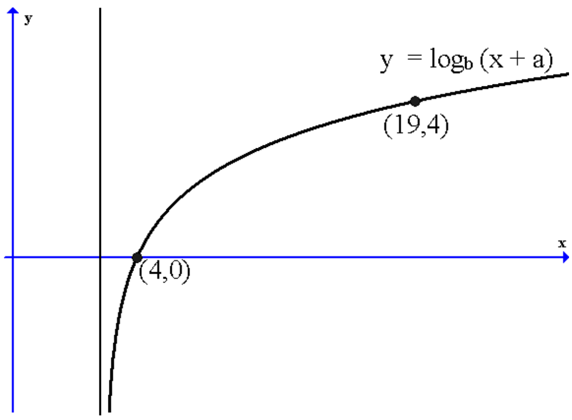
5. The diagram shows part of the graph of $y = \log_a x$.

   (a) Determine the value of a.
   (b) Sketch the graph of $y = \log_a 9x^2$
   (c) Sketch the graph of $y = \log_a \frac{1}{x}$.



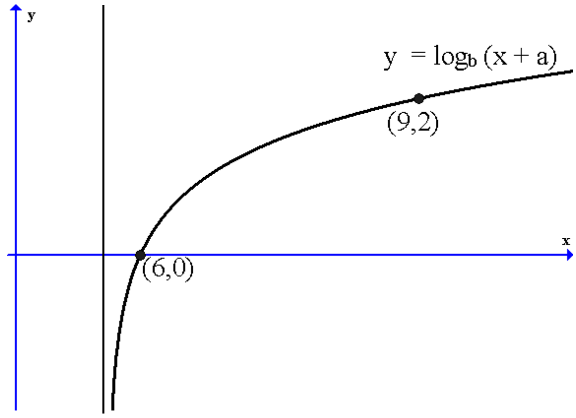
6. The diagram shows the graph of $y = \log_b (x + a)$.

   Find the values of a and b.



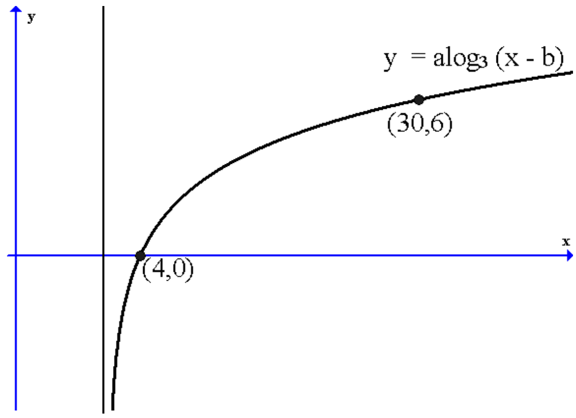
7. The diagram shows the graph of $y = \log_b (x + a)$.

   Find a and b.



8. The diagram shows the graph of $y = a\log_3 (x - b)$.

   Find a and b.



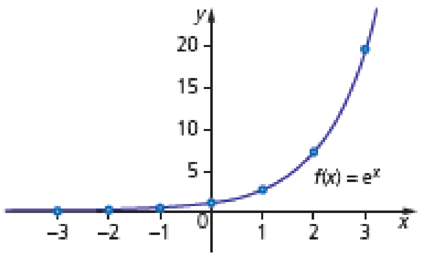
ANSWERS

Exercise 1 :

1 a)

x	-3	-2	-1	0	1	2	3
$f(x)$	0·0	0·1	0·4	1·0	2·7	7·4	20·1

b)



2 a) 2·01
b) 5·47
c) 0·0821
d) 2980
e) 0·301
f) 0·124

Exercise 2 :

1 a) $P_n = 1{\cdot}15^n \times 12000$
b) 27 757

2 a) $V_n = 0{\cdot}88^n \times 25000$
b) £6962·52

3 a) $P_n = 1{\cdot}0125^n \times 6{\cdot}08$ billion
b) 7·417 billion

4 a) $W_h = 0{\cdot}85^h \times 500$
b) 261 ml

5 £1014·59

6 10 years

7 4 hours

8 a) 500
b) 24 701

9 a) 200 mg
b) 134 mg

Exercise 3 :

1. a) $\log_2 8 = 3$ b) $\log_3 9 = 2$
c) $\log_4 64 = 3$ d) $\log_{10} 100 = 2$
e) $\log_3 \frac{1}{9} = -2$ f) $\log_{10} \frac{1}{1000} = -3$
g) $\log_3 1 = 0$ h) $\log_4 4 = 1$.

2. a) $5^2 = 25$ b) $6^2 = 36$ c) $4^3 = 64$
d) $2^{-3} = \frac{1}{8}$ e) $2^{\frac{1}{2}} = \sqrt{2}$ f) $5^{\frac{1}{3}} = \sqrt[3]{5}$
g) $2^0 = 1$ h) $5^1 = 5$.

3. a) $\log_p q = 2$ b) $\log_s r = 3$
c) $\log_b a = c$ d) $\log_k v = t$.

4. a) $3^N = M$ b) $4^t = u$ c) $p^k = m$
d) $b^c = a$.

5. a) 9 b) 64 c) 4 d) 81 e) $\frac{1}{16}$
f) $\frac{1}{9}$ g) $\frac{1}{2}$ h) $\frac{1}{27}$ i) 2 j) 4
k) 3 l) $\sqrt{2}$.

6. a) 3 b) 4 c) 2 d) -2 e) -2
f) -3 g) $\frac{3}{2}$ h) $-\frac{4}{3}$.

Exercise 4 :

1. a) $\log 15$ b) $\log 5$ c) $\log 32$
   d) $\log 28$ e) $\log 4$ f) $\log \frac{3}{2}$
   g) $\log \frac{1}{8}$ h) $\log 2$.
2. a) $\log 12$ b) $\log 8$ c) 0
   d) $\log \frac{8}{9}$ e) $\log \sqrt{5}$ f) $\log 12$.
3. a) $\log_2 6$ b) $\log_3 0{\cdot}6$ c) $\log_2 1{\cdot}5$
   d) $\log_3 \frac{25}{3}$ e) 5 f) 1
   g) $\log_{10} 5$ h) $\log_{10} 500$ i) $\log_{10} 1$
4. a) $\log pq$ b) $\log \frac{r^2}{s}$ c) $\log t\sqrt{u}$
   d) $\log \frac{m}{n}$ e) $\log y\sqrt{x}$ f) $\log \frac{rs^2}{t^3}$.
5. a) $\log_{10} 10a$ b) $\log_{10} \frac{100}{b^3}$
   c) $\log_{10} \frac{a\sqrt{b}}{10}$.
6. a) 5 b) 100 c) 6 d) 12
   e) 3 f) ± 4.

Exercise 5 :

1. a) $3 \log 2$ b) $2 \log 3$
   c) $2 \log 2 + \log 3$ d) $\log 2 + 2 \log 3$
   e) $-\log 2$ f) $-2 \log 3$
   g) $-\log 2 - \log 3$ h) $\log 3 - 2 \log 2$.
2. a) $1 - \log_{10} 2$
   b) (i) $\log_{10} 2 - 1$
   (ii) $2 - 2 \log_{10} 2$
   (iii) $\log_{10} 3 + 1 - \log_{10} 2$
   (iv) $\log_{10} 2 + 1$
   (v) $\log_{10} 3 - 2 \log_{10} 2 + 2$
   (vi) $3\log_{10} 2 - 3$
   (vii) $1 - 5 \log_{10} 2$
   (viii) $\log_{10} 2 + \log_{10} 3 - 1$.
3. $2pq = 1$.
4. $p + q - 2r$.
5. $\log x = \frac{1}{5}(2a + b)$, $\log y = \frac{1}{5}(a - 2b)$.
7. a) $1{\cdot}365$ b) $1{\cdot}771$ c) $0{\cdot}322$.
8. a) $2{\cdot}579$ b) $1{\cdot}654$ c) $-1{\cdot}096$.

Exercise 6 :

1 a) 0·301
b) 0·477
c) 0·699
d) 2·954
e) 0·500
f) −0·097

2 a) 0·693
b) 1·099
c) 1·609
d) 6·802
e) 1·151
f) −0·223

3 a)

	$\log_{10} ab$	$\log_{10} a \times \log_{10} b$	$\log_{10} a + \log_{10} b$
$a = 2, b = 3$	0·778	0·144	0·778
$a = 4, b = 7$	1·447	0·509	1·447
$a = 8, b = 6$	1·681	0·703	1·681

b) $\log_{10} ab = \log_{10} a + \log_{10} b$

4 a)

	$\ln ab$	$\ln a \times \ln b$	$\ln a + \ln b$
$a = 2, b = 3$	1·792	0·762	1·792
$a = 4, b = 7$	3·332	2·698	3·332
$a = 8, b = 6$	3·871	3·726	3·871

b) $\ln ab = \ln a + \ln b$

Exercise 7 :

1 a) $x = 5$
b) $x = 6$
c) $x = 1$
d) $x = 7$

2 a) $x = 6$
b) $x = \frac{7}{8}$
c) $x = \frac{27}{5}$
d) $x = 40$
e) $x = 3$
f) $x = 3$
g) $x = 2$
h) $x = \frac{5}{2}$

3 a) $x = 4$
b) $x = 10$
c) $x = 2$
d) $x = 3$

4 a) $x = 2{\cdot}06$
b) $y = 5{\cdot}13$
c) $a = 1{\cdot}46$
d) $b = 0{\cdot}70$
e) $x = 1{\cdot}29$
f) $m = -0{\cdot}61$

5 a) 2·77
b) 2·12

6 a) $(16, 0)$
b) $(7, 0)$
c) $\left(-\frac{1}{2}, 0\right)$

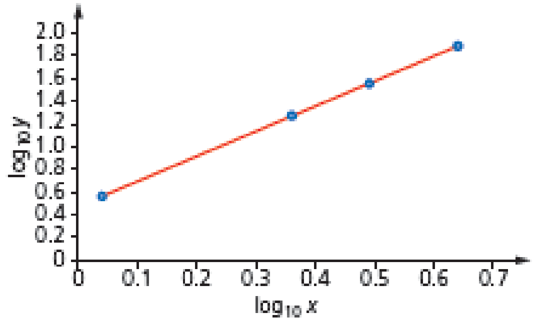
Exercise 8 :

1. a) 12·4 units b) 10 m.
2. a) 2·73 amps b) 3·93 km.
3. a) 41 b) 13·86 years.
4. a) 0·067
   b) $P(4) = 37{\cdot}8 > 30$, (so No).
5. 0·0411 b) 32.2 minutes.
6. a) 0·0719 b) 51·3%.
7. 591 units.

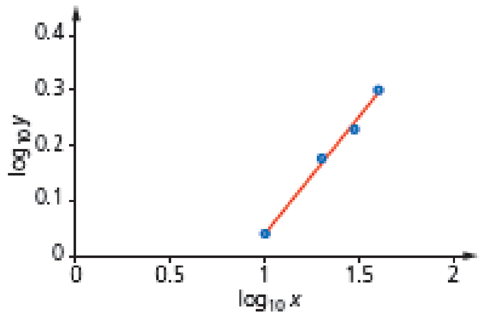
Exercise 9 :

1 **a)** $\ln y = 0{\cdot}75 \ln x + 1{\cdot}5$
b) $a = 4{\cdot}48$ and $b = 0{\cdot}75$

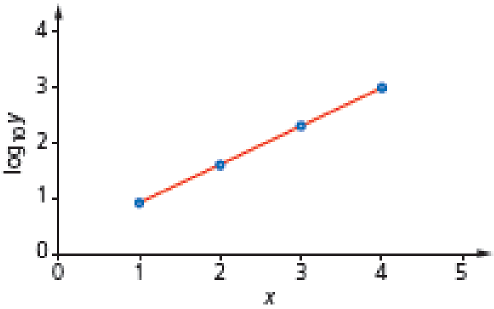
2 $(0, 0{\cdot}7)$
3 **a)** $a = 5$, $b = 6$
b) $a = 31{\cdot}6$, $b = -1{\cdot}5$
4 **a)** $a = 5{\cdot}25$, $b = 2{\cdot}51$
b) $a = 3981$, $b = 0{\cdot}631$
5 **a)**



b) $a = 3$, $b = 2{\cdot}2$
6 **a)**



b) $a = 0{\cdot}41$, $b = 0{\cdot}43$
7 **a)**



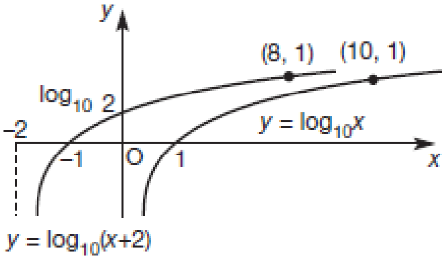
b) $a = 1{\cdot}7$, $b = 5$
8 **a)** $a = \frac{1}{2}$
b) $b = \frac{3}{2}$
c) $\log_{10} 4$
9 **a)** Proof
b) $k = 3{\cdot}17$, $n = 1{\cdot}7$

Exercise 10 :

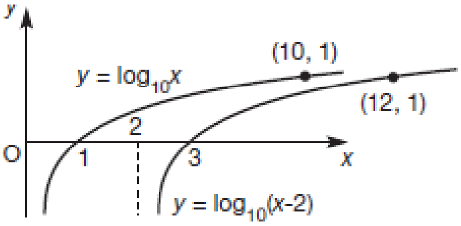
1. a) $Y = 2X$ b) $Y = 3X + 0{\cdot}6$
   c) $Y = X + 0{\cdot}7$.
2. a) $Y = 4X + 1{\cdot}1$ b) $Y = 2{\cdot}3 - X$
   c) $Y = X - 1{\cdot}4$.
3. a) $Y = x + 0{\cdot}3$ b) $Y = 0{\cdot}5x + 0{\cdot}8$
   c) $Y = 0{\cdot}3x + 0{\cdot}7$.
4. a) $Y = x + 1{\cdot}6$ b) $Y = 0{\cdot}7 - 3x$
   c) $Y = 1{\cdot}6x + 2{\cdot}1$.
5. $p = 3\ q^4$.
6. $m = 2{\cdot}7\ v^{3{\cdot}6}$.
7. $s = 0{\cdot}2\ t^{2{\cdot}1}$.
8. $y = 4\ x^{-2}$.
9. $w = 7\ x^{-3}$.
10. $p = 2\ z^{1{\cdot}7}$.
11. $s = 5\ t^4$.
12. $r = 0{\cdot}4\ v^{-2}$.
13. $f = 3{\cdot}2\ 4^t$.
14. $g = 5\ e^{3r}\ [=5 \times (20)^r]$.
15. a) $P = \frac{1}{2}Q + 2$ b) $p = 100\sqrt{q}$.

Exercise 11 :

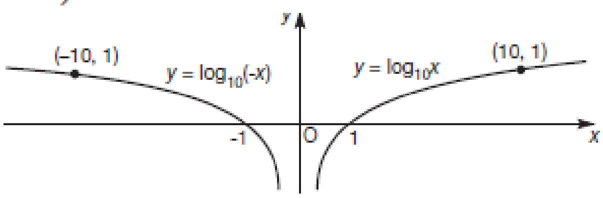
1. a)



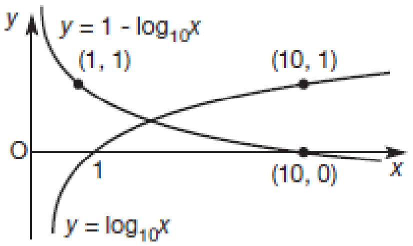
b)



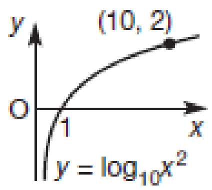
c)



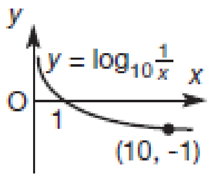
d)



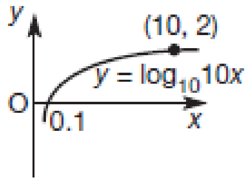
2. a)



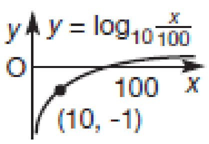
b)



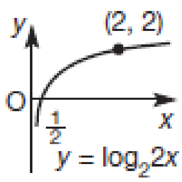
c)



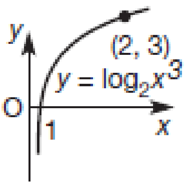
d)



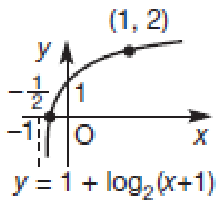
3. a)



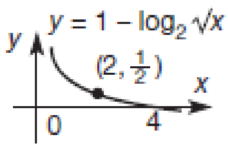
b)



c)



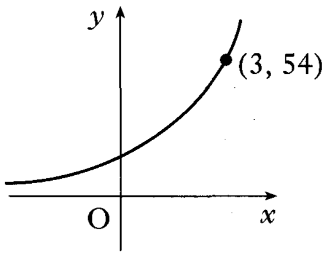
d)



2008 P1

19. The diagram shows part of the graph whose equation is of the form $y = 2m^x$.

What is the value of m?



A 2

B 3

C 8

D 18

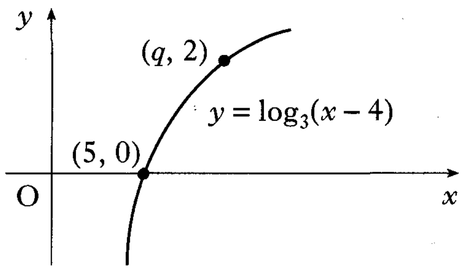
2

Ans B

2008 P1

20. The diagram shows part of the graph of $y = \log_3(x - 4)$.

The point $(q, 2)$ lies on the graph.



What is the value of q?

A 6

B 7

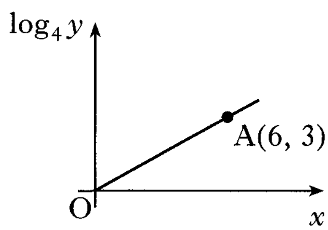
C 8

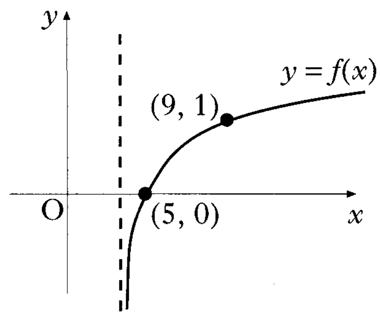
D 13

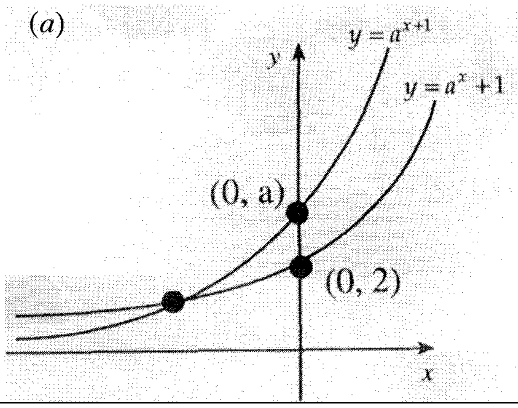
2

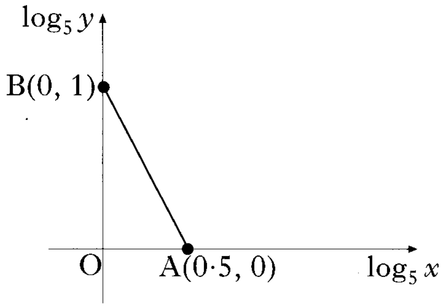
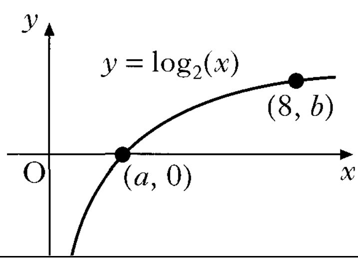
Ans D

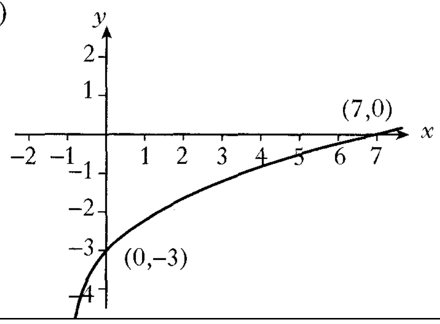
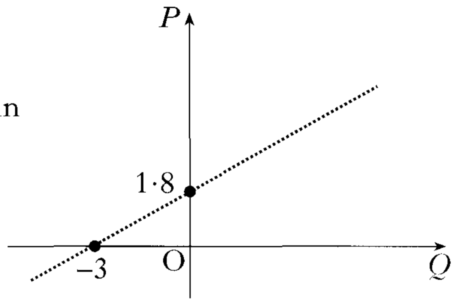
2008 P1	**23.** Functions f, g and h are defined on suitable domains by $f(x) = x^2 - x + 10$, $g(x) = 5 - x$ and $h(x) = \log_2 x$. (a) Find expressions for $h(f(x))$ and $h(g(x))$. (b) Hence solve $h(f(x)) - h(g(x)) = 3$.	3 5
Ans	(a) $h(f(x)) = \log_2(x^2 - x + 10)$ $h(g(x)) = \log_2(5 - x)$ (b) $x = 3, -10$	

2007 P2	**8.** The curve with equation $y = \log_3(x - 1) - 2{\cdot}2$, where $x > 1$, cuts the x-axis at the point $(a, 0)$. Find the value of a.	4
Ans	12.2	
2007 P2	**11.** Two variables x and y satisfy the equation $y = 3 \times 4^x$. (a) Find the value of a if $(a, 6)$ lies on the graph with equation $y = 3 \times 4^x$. (b) If $(-\frac{1}{2}, b)$ also lies on the graph, find b. (c) A graph is drawn of $\log_{10} y$ against x. Show that its equation will be of the form $\log_{10} y = Px + Q$ and state the gradient of this line.	1 1 4
Ans	(a) $a = \frac{1}{2}$ (b) $b = \frac{3}{2}$ (c) $y = 3 \times 4^x$ $\log_{10} y = \log_{10} 3 + \log_{10}(4^x)$ $= \log_{10} 3 + x \log_{10}(4)$ So gradient of line $= \log_{10}(4)$	
2006 P1	**10.** Two variables, x and y, are connected by the law $y = a^x$. The graph of $\log_4 y$ against x is a straight line passing through the origin and the point A(6, 3). Find the value of a.  	4
Ans	$a = 2$	

2006 P2	**11.** It is claimed that a wheel is made from wood which is over 1000 years old. To test this claim, carbon dating is used. The formula $A(t) = A_0e^{-0\cdot000124t}$ is used to determine the age of the wood, where A_0 is the amount of carbon in any living tree, $A(t)$ is the amount of carbon in the wood being dated and t is the age of the wood in years. For the wheel it was found that $A(t)$ was 88% of the amount of carbon in a living tree. Is the claim true?	5
Ans	$t = 1031$ years so claim valid	
2005 P1	**7.** The function f is of the form $f(x) = \log_b (x - a)$. The graph of $y = f(x)$ is shown in the diagram. (a) Write down the values of a and b. (b) State the domain of f.  	2 1
Ans	(a)$a = 4$ $b = 5$ (b)domain is $x > 4$	
2005 P2	**7.** Solve the equation $\log_4(5 - x) - \log_4(3 - x) = 2$, $x < 3$.	4
Ans	$x = \frac{43}{15}$	
2005 P2	**9.** The value V (in £ million) of a cruise ship t years after launch is given by the formula $V = 252e^{-0\cdot06335t}$. (a) What was its value when launched? (b) The owners decide to sell the ship once its value falls below £20 million. After how many years will it be sold?	1 4
Ans	(a) 252(£m) (b)$t = 40$	
2004 P1	**9.** Solve the equation $\log_2(x + 1) - 2\log_2(3) = 3$.	4
Ans	$x = 71$	

2004 P2	**10.** The amount A_t micrograms of a certain radioactive substance remaining after t years decreases according to the formula $A_t = A_0 e^{-0 \cdot 002t}$, where A_0 is the amount present initially. (*a*) If 600 micrograms are left after 1000 years, how many micrograms were present initially? (*b*) The half-life of a substance is the time taken for the amount to decrease to half of its initial amount. What is the half-life of this substance?	 3 4
Ans	(*a*) 4433 (*b*) 347 years	
2003 P1	**12.** Simplify $3\log_e(2e) - 2\log_e(3e)$ expressing your answer in the form $A + \log_e B - \log_e C$ where A, B and C are whole numbers.	4
Ans	$1 + \log_e 8 - \log_e 9$	
2003 P2	**11.** (*a*) (i) Sketch the graph of $y = a^x + 1$, $a > 2$. (ii) On the same diagram, sketch the graph of $y = a^{x+1}$, $a > 2$. (*b*) Prove that the graphs intersect at a point where the x-coordinate is $\log_a\left(\frac{1}{a-1}\right)$.	2 3
Ans	(*a*)  (*b*) $a^{x+1} = a^x + 1$ $a \times a^x - a^x = 1$ $(a-1) \times a^x = 1$ $a^x = \frac{1}{a-1}$ $x = \log_a\left(\frac{1}{a-1}\right)$	
2002W P1	**12.** If $\log_a p = \cos^2 x$ and $\log_a r = \sin^2 x$, show that $pr = a$.	3
Ans	• $\log_a p + \log_a r = \cos^2 x + \sin^2 x$ • $\log_a p + \log_a r = \log_a pr$ • $\log_a pr = 1$ and so $pr = a$ **Alternative** • $p = a^{\cos^2 x}$ $r = a^{\sin^2 x}$ • $pr = a^{\cos^2 x + \sin^2 x}$ • $pr = a^1 = a$	

2002W P2	**9.** A researcher modelled the size N of a colony of bacteria t hours after the beginning of her observations by $N(t) = 950 \times (2{\cdot}6)^{0{\cdot}2t}$. ($a$) What was the size of the colony when observations began? (b) How long does it take for the size of the colony to be multiplied by 10?	1 4
Ans	(a) 950 (b) approx 12 hours	
2002 P1	**11.** The graph illustrates the law $y = kx^n$. If the straight line passes through A(0·5, 0) and B(0, 1), find the values of k and n. 	4
Ans	$\log_5 y = -2\,(\log_5 x) + 1$ $\log_5 y = \log_5 x^{-2} + \log_5 5$ $\log_5 y = \log_5 5x^{-2}$ $k = 5,\ n = -2$ $(y = 5x^{-2})$	
2002 P2	**7.** Find the x-coordinate of the point where the graph of the curve with equation $y = \log_3(x-2) + 1$ intersects the x-axis.	3
Ans	$\log_3(x-2) = -1$ $x = 2\frac{1}{3}$	
2001 P1	**8.** Find x if $4\log_x 6 - 2\log_x 4 = 1$.	3
Ans	*x = 81*	
2001 P1	**10.** The diagram shows a sketch of part of the graph of $y = \log_2(x)$. (a) State the values of a and b. (b) Sketch the graph of $y = \log_2(x+1) - 3$. 	1 3

Ans	(*a*) $a = 1,\ b = 3$ (*b*) 	
2001 P2	**9.** Before a forest fire was brought under control, the spread of the fire was described by a law of the form $A = A_0e^{kt}$ where A_0 is the area covered by the fire when it was first detected and A is the area covered by the fire t hours later. If it takes one and half hours for the area of the forest fire to double, find the value of the constant k.	3
Ans	$k = 0{\cdot}46$	
2000 P1	**9.** Evaluate $\log_5 2 + \log_5 50 - \log_5 4$.	3
Ans	2	
2000 P2	**11.** The results of an experiment give rise to the graph shown. (*a*) Write down the equation of the line in terms of P and Q.  It is given that $P = \log_e p$ and $Q = \log_e q$. (*b*) Show that p and q satisfy a relationship of the form $p = aq^b$, stating the values of a and b.	2 4
Ans	(*a*) $P = 0{\cdot}6Q + 1{\cdot}8$ (*b*) $a = 6{\cdot}05,\ b = 0{\cdot}6$	
Specimen 2 P2	**7.** The intensity I_t of light is reduced as it passes through a filter according to the law $I_t = I_0e^{-kt}$ where I_0 is the initial intensity and I_t is the intensity after passing through a filter of thickness t cm. k is a constant. (*a*) A filter of thickness 4 cm reduces the intensity from 120 candle-power to 90 candle-power. Find the value of k. (*b*) The light is passed through a filter of thickness 10 cm. Find the percentage reduction in its intensity.	4 3

Ans

(a) $90 = 120e^{-4k} \Rightarrow e^{-4k} = 0{\cdot}75$

$-4k = \ln 0{\cdot}75 \Rightarrow k = 0{\cdot}0719$

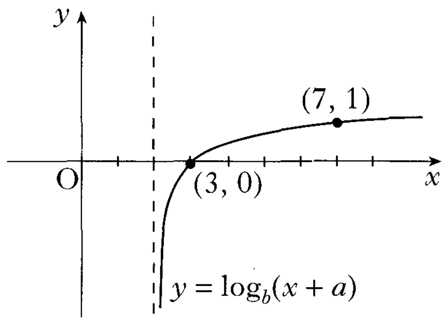
(b) $I_{10} = I_0 e^{-10\times 0\cdot 0719}$

$\dfrac{I_{10}}{I_0} = 0\cdot 487$ so a $51{\cdot}3\%$ reduction

Specimen 2 P2

8. The diagram shows part of the graph of $y = \log_b(x + a)$.

Determine the values of a and b. 3



Ans

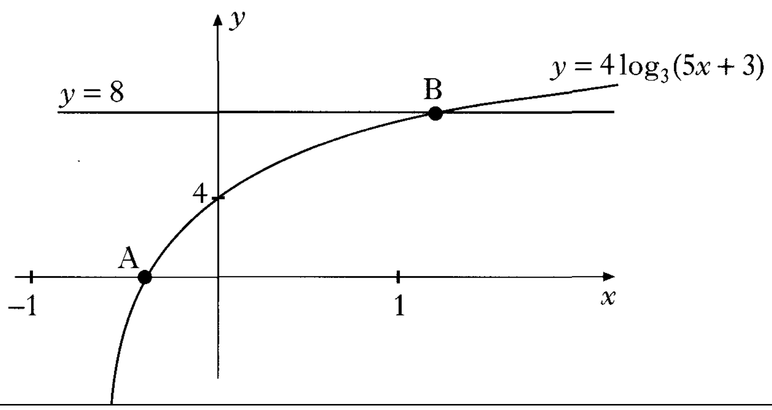
translation $\Rightarrow a = -2$

$(7,1) \Rightarrow 1 = \log_b(7 - 2) \Rightarrow b = 5$

Specimen 1 P1

10. Part of the graph of $y = 4\log_3(5x + 3)$ is shown in the diagram. This graph crosses the x-axis at the point A and the straight line $y = 8$ at the point B.

Find the x-coordinate of B. 3



Ans

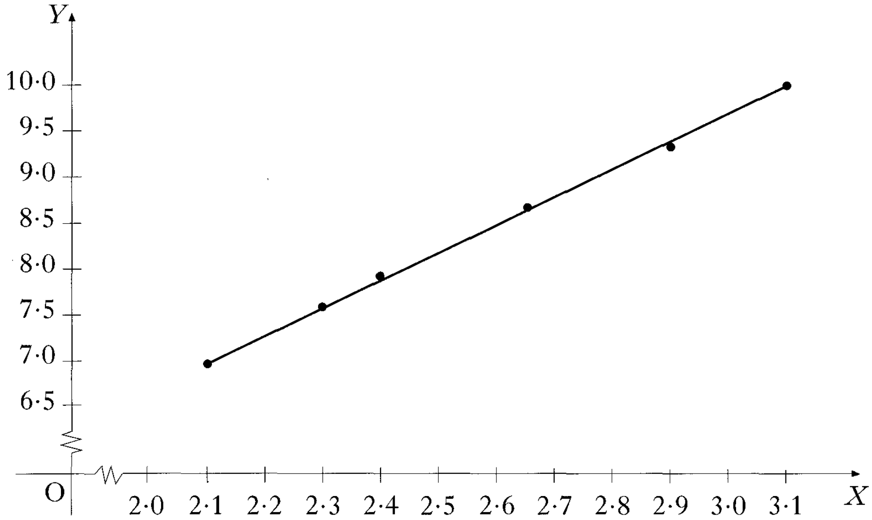
$\frac{6}{5}$

Specimen 1 P2	**10.** Six spherical sponges were dipped in water and weighed to see how much water each could absorb. The diameter (x millimetres) and the gain in weight (y grams) were measured and recorded for each sponge. It is thought that x and y are connected by a relationship of the form $y = ax^b$.	

By taking logarithms of the values of x and y, the table below was constructed.

$X\ (= \log_e x)$	2·10	2·31	2·40	2·65	2·90	3·10
$Y\ (= \log_e y)$	7·00	7·60	7·92	8·70	9·38	10·00

A graph was drawn and is shown below.



Find the equation of the line in the form $Y = mX + c$.

3

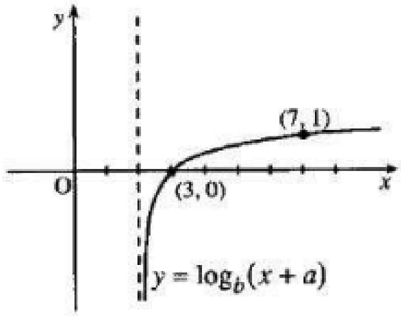
Ans $Y = 3X + 0·7$

Higher Mathematics
Logs and Exponentials Homework

1. Evaluate $\log_5 2 + \log_5 50 - \log_5 4$. **3**

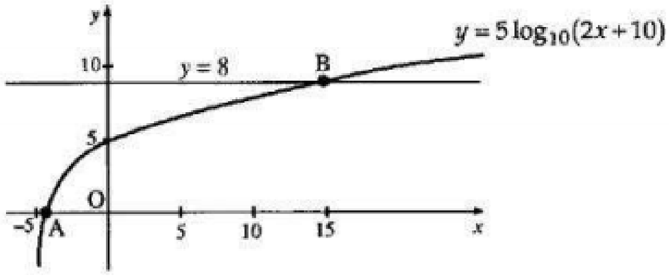
2. Given $x = \log_5 3 + \log_5 4$, find algebraically the value of x. **4**

3. The diagram shows part of the graph of $y = \log_b(x+a)$.
Determine the values of a and b. 3



4. Part of the graph of $y = 5\log_{10}(2x+10)$ is shown in the diagram. This graph crosses the x-axis at the point A and the straight line $y = 8$ at the point B.

Find algebraically the x-coordinates of A and B. 4



5. (*a*) A tractor tyre is inflated to a pressure of 50 units.
Twenty-four hours later the pressure has dropped to 10 units.

If the pressure, P_t units, after t hours is given by the formula $P_t = P_0 e^{-kt}$, find the value of k, to three decimal places. **(5)**

(*b*) The tyre manufacturer advises that serious damage to the tyre will result if it is used when the pressure drops below 30 units.

If the farmer inflates the tyre to 50 units and drives the tractor for four hours, can the tractor be driven further without inflating the tyre and without risking serious damage to the tyre? **(4)**